Introduction
High order approximation schemes are often required in the numerical analysis of nonlocal problems. However, several issues arise from the implementation of either computationally intensive algorithms or highly continuous approximation schemes. In this research, non-conventional approximation techniques are tested on nonlocal elastic boundary value problems.

Higher order approximations
Beside classical Hermitian and Lagrangian finite element discretizations, we investigate B-spline finite elements and \( C^\infty \) generalized finite elements (\( C^\infty \) GFEM). The former are built with highly continuous piecewise polynomial basis functions (Figure 1a), whereas the latter (Figure 1b) make use of a \( C^\infty \) Partition of Unity (PoU), supported on a standard finite element grid, and polynomial enrichments of arbitrary degree [1].

![Figure 1: (a) B-spline basis functions and (b) \( C^\infty \) GFEM PoU](image)

Nonlocal constitutive models
The mechanical response of one-dimensional tensile rods is studied by means of two nonlocal formulations. In the integro-differential formulation, the stress \( \sigma \) at a given point is expressed as a weighted function of the strain \( \epsilon \) at surrounding points according to

\[
\sigma(x) = E\xi_1\epsilon(x) + \frac{E\xi_2}{2\ell} \int_\Omega e^{-\frac{|x-\tilde{x}|}{\ell^3}}\epsilon(\tilde{x})\,d\tilde{x},
\]

where \( \xi_1 \) and \( \xi_2 \) are constitutive parameters such that \( \xi_1 + \xi_2 = 1 \) with \( \xi_1 \geq 1 \). \( E \) is the Young’s modulus and \( \ell \) the material characteristic length. In the strain-gradient formulation, the classical elasticity constitutive equation is augmented with the Laplacian of the strain as in

\[
\sigma(x) = E \left( \epsilon(x) - g^2 \nabla^2 \epsilon(x) \right),
\]

where \( g \) is the length scale parameter related to the volumetric elastic strain energy.

Results
The numerical results depend on both the approximation scheme used and the constitutive assumptions. As indicated in Figure 2, where only half domain is shown due to symmetry, the integro-differential formulation leads to spurious oscillations along the whole domain. In contrast, despite the analytical solution being the same [2], in the strain-gradient model the oscillations are limited to the boundaries. Furthermore, \( C^\infty \) GFEM and, in general, Hermite finite elements are not suitable when the continuity of the exact solution decreases at a prescribed point. Arbitrarily continuous B-spline basis functions easily overcome this problem by means of multiple knots insertion (Figure 4).

In general, the accuracy of the numerical results increases considerably with the order of the approximation. Among the basis functions considered here, B-spline basis functions are the most efficient. Furthermore, higher order discretizations fully comply with the continuity requirements imposed by the strain gradient formulation.

![Figure 2: Homogeneous rod (integro-differential model)](image)

![Figure 3: Homogeneous rod (strain-gradient model)](image)

![Figure 4: Bimaterial rod (strain-gradient model)](image)

References